Chapter 3:
Determinants and Diagonalization

Sec. 3.1:
The Cofactor Expansion

What is a Determinant?

If you start off with a square matrix A...

- The determinant of this matrix is a number
- Notations for the determinant of A: $\det(A)$, $|A|$
- It's main uses…

1) it's a quick way to see if a matrix has an inverse

$$|A| \neq 0 \quad \text{iff} \quad A \text{ has an inverse}$$

2) gives a way to find the inverse of a matrix

$$\text{If } |A| \neq 0, \text{ then } A^{-1} = \frac{1}{|A|} adj(A)$$

How do you Calculate a Determinant?

1×1 case

For a 1×1 matrix, its determinant is just the number in the matrix.

<u>Ex 1</u>:

a) For the matrix $A = [9]$, find $\det(A)$

b) Find $|-4|$

How do you Calculate a Determinant?

2×2 case

For a 2×2 matrix, the formula is…

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

<u>Ex 2</u>: Find $\begin{vmatrix} 4 & -3 \\ -5 & 2 \end{vmatrix}$

How do you Calculate a Determinant?

3×3 case

Ex 3: Use the criss-cross method to find $\begin{vmatrix} 5 & 2 & -1 \\ -3 & 4 & 0 \\ -1 & 6 & 2 \end{vmatrix}$

How do you Calculate a Determinant?
$n \times n$ case for $n \geq 3$

The Cofactor Expansion…

Notation: If A is an $n \times n$ matrix…

- a_{ij} is the number in the ith row and jth column of the matrix A
- A_{ij} is the matrix obtained by deleting the ith row and jth column of the matrix A
- $M_{ij} \equiv \det(A_{ij})$ are called the minors of matrix A
- $c_{ij}(A) \equiv (-1)^{i+j} M_{ij}$ or $c_{ij}(A) \equiv (-1)^{i+j} \det(A_{ij})$ are called the cofactors of the matrix A

How do you Calculate a Determinant?

$n \times n$ case for $n \geq 3$

The Cofactor Expansion…

Ex 4: If $A = \begin{bmatrix} -2 & 7 & 1 \\ 4 & 0 & 9 \\ 2 & 2 & 3 \end{bmatrix}$, find… a) a_{23} b) A_{23} c) M_{23} d) $c_{23}(A)$

How do you Calculate a Determinant?
$n \times n$ case for $n \geq 3$

The Cofactor Expansion…

1. Suppose A is an $n \times n$ matrix
2. Pick any row or column to expand along
3. Multiply each number in this row or column by its cofactor
4. Add the numbers from step 3 to get the determinant of A

<u>Notes</u>:

- If you expand along row i then…
$$|A| = a_{i1}c_{i1}(A) + a_{i2}c_{i2}(A) + \cdots a_{in}c_{in}(A)$$
- If you expand along column j then…
$$|A| = a_{1j}c_{1j}(A) + a_{2j}c_{2j}(A) + \cdots a_{nj}c_{nj}(A)$$
- You will always get the same answer for the determinant no matter which row or column you choose to expand along

How do you Calculate a Determinant?

$n \times n$ case for $n \geq 3$

Ex 5: Use the cofactor expansion method to find $\begin{vmatrix} 3 & 0 & -1 \\ 2 & 7 & 0 \\ 5 & 0 & 4 \end{vmatrix}$ twice.

How do you Calculate a Determinant?

$n \times n$ case for $n \geq 3$

Ex 6: Use the cofactor expansion method to find $\begin{vmatrix} 7 & 5 & 3 & 2 \\ -2 & 8 & 1 & 9 \\ 0 & 4 & 6 & -8 \\ 9 & 8 & 4 & 3 \end{vmatrix}$

How do you Calculate a Determinant?
$n \times n$ case for $n \geq 3$

The Cofactor Expansion…

<u>Notes</u>:

- You will always get the same answer for the determinant no matter which row or column you choose to expand along
- Using the cofactor expansion to find a determinant turns an $n \times n$ determinant into a sum of n $(n-1) \times (n-1)$ determinants
- Choosing a row or column that has many 0's in it will cut down the work in finding a determinant

How do you Calculate a Determinant? On TI-83/84

Ex 7: Use a TI-83/84 to find

$$\begin{vmatrix} 4 & 1 & 5 \\ -2 & -3 & 1 \\ 3 & 9 & 7 \end{vmatrix}$$

How Elementary Row Operations Affect The Determinant

Suppose B is the matrix obtained when a single elementary row operation is done to a square matrix A.

$$A \xrightarrow{\text{Elementary Row Operation}} B$$

1) If B is obtained by interchanging 2 rows of A, then

$$|B| = -|A|$$

2) If B is obtained by multiplying a row of A by a number c, then

$$|B| = c|A|$$

3) If B is obtained by multiplying a row of A by a number then adding that row to a different row, then

$$|B| = |A|$$

How Elementary Row Operations Affect The Determinant

$$A \xrightarrow{\substack{\text{Elementary} \\ \text{Row} \\ \text{Operation}}} B$$

(Understand/Verify/Prove)

1) If B is obtained by interchanging 2 rows of A, then $|B| = -|A|$

How Elementary Row Operations Affect The Determinant

$$A \xrightarrow[\text{}]{\substack{\text{Elementary}\\ \text{Row}\\ \text{Operation}}} B$$

(Understand/Verify/Prove)

2) If B is obtained by multiplying a row of A by a number c, then $|B| = c|A|$

How Elementary Row Operations Affect The Determinant

$$A \xrightarrow[]{\text{Elementary Row Operation}} B$$

(Understand/Verify/Prove)

3) If B is obtained by multiplying a row of A by a number then adding that row to a different row, then $|B| = |A|$

(To prove this, you have to first prove that if 2 rows in a matrix are identical, then det = 0)

How Elementary Row Operations Affect The Determinant

Def:

1) A square matrix is upper triangular if all entries below its main diagonal are 0.

Like $\begin{bmatrix} 7 & 1 & 3 \\ 0 & 2 & 9 \\ 0 & 0 & -3 \end{bmatrix}$

2) A square matrix is lower triangular if all entries above its main diagonal are 0.

Like $\begin{bmatrix} -2 & 0 & 0 \\ 5 & 0 & 0 \\ 9 & -4 & 8 \end{bmatrix}$

How Elementary Row Operations Affect The Determinant

Def:

3) A square matrix is triangular if it is either upper triangular or lower triangular.

How Elementary Row Operations Affect The Determinant

Ex 8: Find the following determinant by first reducing it to upper triangular form.

$$\begin{vmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{vmatrix}$$

How Elementary Row Operations Affect The Determinant

Ex 8: Find the following determinant by first reducing it to upper triangular form.

$$\begin{vmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & -1 & 2 \end{vmatrix}$$

Theorem 3.1.4

If A is a square triangular matrix, then det A is the product of the entries on the main diagonal.

How Elementary Row Operations Affect The Determinant

Ex 9:

If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 4$, compute $\begin{vmatrix} -2a & -2b & -2c \\ 2p+x & 2q+y & 2r+z \\ 3x & 3y & 3z \end{vmatrix}$

Other Facts About Determinant / Theorems

Theorem 3.1.1: Cofactor Expansion Theorem[2]

The determinant of an $n \times n$ matrix A can be computed by using the cofactor expansion along any row or column of A. That is $\det A$ can be computed by multiplying each entry of the row or column by the corresponding cofactor and adding the results.

Theorem 3.1.2

Let A denote an $n \times n$ matrix.

1. *If A has a row or column of zeros, $\det A = 0$.*
2. *If two distinct rows (or columns) of A are interchanged, the determinant of the resulting matrix is $-\det A$.*
3. *If a row (or column) of A is multiplied by a constant u, the determinant of the resulting matrix is $u(\det A)$.*
4. *If two distinct rows (or columns) of A are identical, $\det A = 0$.*
5. *If a multiple of one row of A is added to a different row (or if a multiple of a column is added to a different column), the determinant of the resulting matrix is $\det A$.*

Also, if a row (or column) of A is a multiple of another row (or column) of A, then $\det(A) = 0$.

Other Facts About Determinant / Theorems

Ex 10: Find the determinants below quickly by inspection and state why the determinant is as you say.

a) $\begin{vmatrix} 4 & 1 & 0 & 9 \\ 0 & -3 & 0 & 12 \\ -2 & 1 & 0 & 4 \\ 6 & 2 & 0 & 1 \end{vmatrix}$ b) $\begin{vmatrix} 0 & -1 & 7 & 9 \\ 1 & 2 & 3 & 4 \\ 8 & 2 & 7 & 5 \\ 1 & 2 & 3 & 4 \end{vmatrix}$ c) $\begin{vmatrix} 2 & 1 & -4 \\ 7 & 0 & -3 \\ -6 & -3 & 12 \end{vmatrix}$

Other Facts About Determinant / Theorems

Theorem 3.1.3

If A is an $n \times n$ *matrix, then* $\det(uA) = u^n \det A$ *for any number* u.

Ex 11:

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{bmatrix}$. Then $\det(A) = 24$. Find $\det(2A)$.

Are These True About Determinants?

<u>Book Exercise 3.1.9</u>:

In the following, assume A and B are square matrices. In each case, prove the following or give a counter example showing that it is false.

a) $\det(A+B) = \det(A) + \det(B)$

b) If $\det(A) = 0$, then A has 2 equal rows

c) If A is 2×2, then $\det(A^T) = \det(A)$

d) If R is reduced row-echelon form of A, then $\det(A) = \det(R)$

e) If A is 2×2, then $\det(7A) = 49\det(A)$

f) $\det(A^T) = -\det(A)$

g) $\det(-A) = -\det(A)$

h) If $\det(A) = \det(B)$ where A and B are the same size, then $A = B$.

What you need to know from the book

Book reading

Pages: 145-154 (only top portion up until Thm. 3.1.4).

Problems you need to know how to do from the book

#'s 1-10, 13-18, 20, 22-26